MATHEMATICS

ON THE VOLUME OF COMPOUND CONVEX BODIES

BY

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Recently MAHLER [1, 2] developed a number-geometrical theory of compound convex bodies. One of the problems he dealt with was to obtain estimates for the volume of the so-called compound of a given number of convex bodies. In this note I shall give a further contribution to this problem.

Let $1 \leqq p \leqq n-1$ and $N=\binom{n}{p}$, and let $K^{(1)}, K^{(2)}, \ldots, K^{(p)}$ be any p bounded closed convex bodies in R_n, symmetric about the origin. Then the *compound* of these p bodies, denoted by K, is defined as follows. For any p points

$$X^{(\pi)}=(x_{\pi 1}, x_{\pi 2}, \ldots, x_{\pi n}) \qquad (\pi=1, 2, \ldots, p)$$

in R_n let $[X^{(1)}, X^{(2)}, \ldots, X^{(p)}]$ denote the point (vector) in R_N whose coordinates are given by the N determinants

$$(1) \qquad x_{\nu_1\nu_2\ldots\nu_p}=\begin{vmatrix} x_{1\nu_1} & x_{1\nu_2} & \ldots & x_{1\nu_p} \\ x_{2\nu_1} & x_{2\nu_2} & \ldots & x_{2\nu_p} \\ \cdot & \cdot & \cdot & \cdot \\ x_{p\nu_1} & x_{p\nu_2} & \ldots & x_{p\nu_p} \end{vmatrix} \quad (1 \leqq \nu_1 < \nu_2 < \ldots < \nu_p \leqq n),$$

taken in some definite order. Then K is the convex hull of the set of points

$$\Xi=[X^{(1)}, X^{(2)}, \ldots, X^{(p)}] \text{ with } X^{(\pi)} \in K^{(\pi)} \qquad (\pi=1, 2, \ldots, p).$$

We further write $P=\binom{n-1}{p-1}$ and put

$$(2) \qquad Q=V(\mathrm{K})\left\{\prod_{\pi=1}^{p} V(K^{(\pi)})\right\}^{-P/p}.$$

MAHLER [1] proved that, if the bodies $K^{(\pi)}$ are identical, the quotient Q has positive upper and lower bounds which only depend on n and p, and gave interesting applications of this result. He further showed that in the general case of nonidentical bodies there is no such upper bound for Q. On the other hand, he established the existence of a positive lower bound for Q only depending on n and p in the case that the bodies $K^{(\pi)}$ fall into two classes of identical bodies [2]. Here I shall deduce the following general

Theorem. *There exists a positive constant c only depending on n and p, such that always $Q > c$.*

In fact I shall prove that one can take $c=1/N!$.

In the following we shall denote by $m_1^{(\pi)}, m_2^{(\pi)}, \ldots, m_n^{(\pi)}$ the successive minima of $K^{(\pi)}$ with respect to the lattice of points with integral coordinates $(\pi=1, 2, \ldots, p)$. We shall make use of Minkowski's well-known inequality for the successive minima of a convex body, according to which we have

$$\{\prod_{\nu=1}^{n} m_\nu^{(\pi)}\} \cdot V(K^{(\pi)}) \leqq 2^n \qquad (\pi=1, 2, \ldots, p). \tag{3}$$

Further, for $\pi=1, 2, \ldots, p$ let

$$A^{(\pi,\nu)}=(a_1^{(\pi,\nu)}, a_2^{(\pi,\nu)}, \ldots, a_n^{(\pi,\nu)}) \qquad (\nu=1, 2, \ldots, n)$$

be n points with integral coordinates such that $A^{(\pi,1)}, A^{(\pi,2)}, \ldots, A^{(\pi,n)}$ are independent and that

$$A^{(\pi,\nu)} \in m_\nu^{(\pi)} K^{(\pi)} \qquad (\nu=1, 2, \ldots, n). \tag{4}$$

For our purposes, it is convenient to arrange the determinants (1) in lexicographical order. We arrange in the same order the sets of integers $(\nu_1, \nu_2, \ldots, \nu_p)$, where $1 \leqq \nu_1 < \nu_2 < \ldots < \nu_p \leqq n$, and in this order denote these sets by

$$\{\nu_{1,i}, \nu_{2,i}, \ldots, \nu_{p,i}\} \qquad (i=1, 2, \ldots, N).$$

Thus, if $1 \leqq i < j \leqq N$ and π is the lowest index with $\nu_{\pi,i} \neq \nu_{\pi,j}$, we have $\nu_{\pi,i} < \nu_{\pi,j}$.

An arbitrary vector $[X^{(1)}, X^{(2)}, \ldots, X^{(p)}]$, where $X^{(\pi)}=(x_{\pi 1}, x_{\pi 2}, \ldots, x_{\pi n})$ $(\pi=1, 2, \ldots, p)$ can be broken up into $n-p+1$ projections as follows. Let the first projection be built up from the first $\binom{n-1}{p-1}$ components, i.e. the quantities (1) with $\nu_1=1$. Similarly, let the second projection consist of the next $\binom{n-2}{p-1}$ components, i.e. the quantities (1) with $\nu_1=2$; generally, let the qth projection consist of the quantities (1) with

$$\nu_1=q \ (q=1, 2, \ldots, n-p+1).$$

We shall denote the linear subspaces of R_N, in which these projections lie and which successively have dimensions $\binom{n-1}{p-1}, \binom{n-2}{p-1}, \ldots, \binom{p-1}{p-1}$, by $R^{(1)}, R^{(2)}, \ldots, R^{(n-p+1)}$ respectively. The values of these dimensions are in accordance with the formula

$$\binom{n-1}{p-1}+\binom{n-2}{p-1}+\ldots+\binom{p-1}{p-1}=\binom{n}{p}.$$

In the course of the proof of our theorem we shall choose in a suitable way N sets of p points, one in each $K^{(\pi)}$, such that the N corresponding points in K are independent and determine a polyhedron whose volume has the required order of magnitude. The following lemma is essential.

Lemma. *Let p, n, m be positive integers with $1 \leqq p \leqq n \leqq m$, and let*

$m=n+\varrho$. Let there be given p systems of m vectors

$$B^{(\pi,\mu)}=(b_1^{(\pi,\mu)}, b_2^{(\pi,\mu)}, \ldots, b_n^{(\pi,\mu)})$$

($\mu=1, 2, \ldots, m$; $\pi=1, 2, \ldots, p$) and suppose that for $\pi=1, 2, \ldots, p$ the matrix formed by the vectors $B^{(\pi,1)}, B^{(\pi,2)}, \ldots, B^{(\pi,m)}$ has exactly rank n. Let the positive integers $\nu_{\pi,i}$ be defined as above ($\pi=1, 2, \ldots, p$; $i=1, 2, \ldots, N$). Then there exist N sets of p positive integers $\leqq m$,

$$(5) \qquad \{\mu_{1,i}, \mu_{2,i}, \ldots, \mu_{p,i}\} \quad (i=1, 2, \ldots, N)$$

say, such that

1. $\mu_{\pi,i} \leqq \nu_{\pi,i}+\varrho$ for $\pi=1, 2, \ldots, p$ and $i=1, 2, \ldots, N$
2. *the N vectors $[B^{(1,\mu_{1,i})}, B^{(2,\mu_{2,i})}, \ldots, B^{(p,\mu_{p,i})}]$*

in the space R_N are linearly independent.

Proof. We first remark that, if the assertions of the lemma hold for some set of vectors $B^{(\pi,\mu)}$, they remain true for each other set of pm vectors obtained by subjecting all vectors $B^{(\pi,\mu)}$ to a non-singular transformation Ω of R_n, with the same sets (5). For then the vectors entering in 2 are all subjected to the adjoint transformation of R_N, which likewise is non-singular (see MAHLER [1]).

The proof of the lemma is by induction on p. The lemma is trivially true for $p=1$, since then R_N coincides with R_n and we need only to take n positive integers $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)}$ with increasing order of magnitude, such that the vectors $B^{(1,\mu^{(i)})}$ are independent. Now let p be a fixed integer >1 and suppose that the lemma is true with $p-1$ instead of p (and arbitrary n, m).

In virtue of the hypotheses of our lemma we can choose n positive integers $\mu_1, \mu_2, \ldots, \mu_n$ with $1 \leqq \mu_1 < \mu_2 \ldots < \mu_n \leqq m$, such that the vectors $B^{(1,\mu_1)}, B^{(1,\mu_2)}, \ldots, B^{(1,\mu_n)}$ are linearly independent. By the above remark it is no loss of generality to suppose that these vectors are the unit vectors in R_n, i.e. that

$$B^{(1,\mu_1)}=(1, 0, 0, \ldots, 0),\ B^{(1,\mu_2)}=(0, 1, 0, \ldots, 0),$$
$$\ldots, B^{(1,\mu_n)}=(0, 0, 0, \ldots, 1).$$

We now consider the points

$$(6) \qquad [B^{(1,\mu_1)}, B^{(2,m_2)}, B^{(3,m_3)}, \ldots, B^{(p,m_p)}]$$

in R_N, where $m_2, m_3, \ldots, m_p$ are arbitrary positive integers $\leqq m$. Since $B^{(1,\mu_1)}=(1, 0, 0, \ldots, 0)$, these points all lie in the subspace $R^{(1)}$ and, as points of $R^{(1)}$, they have the form

$$[\bar{B}^{(2,m_2)}, \bar{B}^{(3,m_3)}, \ldots, \bar{B}^{(p,m_p)}],$$

where we have put

$$\bar{B}^{(\pi,\mu)}=(b_2^{(\pi,\mu)}, b_3^{(\pi,\mu)}, \ldots, b_n^{(\pi,\mu)})$$

$(\pi=2, 3, \ldots, p;\ \mu=1, 2, \ldots, m)$. Clearly for fixed π the vectors $\bar{B}^{(\pi,1)}, \bar{B}^{(\pi,2)}, \ldots$ $\ldots, \bar{B}^{(\pi,m)}$ form a matrix of rank $n-1$. Further the sets $\{\nu_{2,i}, \nu_{3,i}, \ldots, \nu_{p,i}\}$ with $i=1, 2, \ldots, \binom{n-1}{p-1}$ are just those for which $2 \leqq \nu_{2,i} < \nu_{3,1} < \ldots < \nu_{p,1} \leqq n$. So, by our induction hypothesis, there are $\binom{n-1}{p-1}$ sets of $p-1$ positive integers $\leqq m$, $\{\mu_{2,i}, \mu_{3,i}, \ldots, \mu_{p,i}\}$ say, such that the vectors

$$[\bar{B}^{(2,\mu_{2,i})}, \bar{B}^{(3,\mu_{3,i})}, \ldots, \bar{B}^{(p,\mu_{p,i})}]$$

are linearly independent and that $\mu_{\pi,i} \leqq (\nu_{\pi,i}-1)+(\varrho+1)=\nu_{\pi,i}+\varrho$ for $\pi=2, 3, \ldots, p$ and $i=1, 2, \ldots, \binom{n-1}{p-1}$. We finally take $\mu_{1,i}=\mu_1$ for $i=1, 2, \ldots, \binom{n-1}{p-1}$. Then the vectors $[B^{(1,\mu_{1,i})}, B^{(2,\mu_{2,i})}, \ldots, B^{(p,\mu_{p,i})}]$ with $i=1, 2, \ldots, \binom{n-1}{p-1}$ are linearly independent and are all lying in $R^{(1)}$, whereas $\mu_{\pi,i} \leqq \nu_{\pi,i}+\varrho$ for $\pi=1, 2, \ldots, p$.

More generally, we consider the points

$$[B^{(1,\mu_q)}, B^{(2,m_2)}, B^{(3,m_3)}, \ldots, B^{(p,m_p)}] \tag{7}$$

with arbitrary positive integers $m_2, m_3, \ldots, m_p \leqq m$ and a fixed positive integer $q \leqq n-p+1$. These points are all lying in the linear subspace of R_N, which is the direct sum of the subspaces $R^{(1)}, R^{(2)}, \ldots, R^{(q)}$ (actually only certain $\binom{n-1}{p-1}$ coordinates of these points can differ from zero). Now the determinants (1), for which $\nu_1=q$, are just those coordinates of the point $[X^{(1)}, X^{(2)}, \ldots, X^{(p)}]$ which make up the subspace $R^{(q)}$. Hence, since $B^{(1,\mu_q)}$ is the qth unit point, the projections of the points (7) on $R^{(q)}$ have the form

$$[\bar{B}^{(2,m_2)}, \bar{B}^{(3,m_3)}, \ldots, \bar{B}^{(p,m_p)}],$$

where now $\bar{B}^{(\pi,\mu)}=(b_{q+1}^{(\pi,\mu)}, b_{q+2}^{(\pi,\mu)}, \ldots, b_n^{(\pi,\mu)})$ $(\pi=2, 3, \ldots, p;\ \mu=1, 2, \ldots, m)$. Clearly for fixed π the vectors

$$[\bar{B}^{(\pi,1)}, \bar{B}^{(\pi,2)}, \ldots, \bar{B}^{(\pi,m)}]$$

form a matrix of rank $n-q$. Further the sets $\mu_{2,i}, \mu_{3,i}, \ldots, \mu_{p,i}$ with

$$i=\binom{n-1}{p-1}+\binom{n-2}{p-1}+\ldots+\binom{n-q-1}{p-1}+i_0, \quad 1 \leqq i_0 \leqq \binom{n-q}{p-1}$$

are just those for which

$$\nu_{1,i}=q,\ q+1 \leqq \nu_{2,i} < \nu_{3,i} < \ldots < \nu_{p,i} \leqq n.$$

So, by our induction hypothesis, there are $\binom{n-1}{p-1}$ sets of $p-1$ positive integers $\leqq m$, $\{\mu_{2,i}, \mu_{3,i}, \ldots, \mu_{p,i}\}$ say, such that the vectors

$$[\bar{B}^{(2,\mu_{2,i})}, \bar{B}^{(3,\mu_{3,i})}, \ldots, \bar{B}^{(p,\mu_{p,i})}]$$

are linearly independent and that $\mu_{\pi,i} \leqq (\nu_{\pi,i} - q) + (\varrho + q) = \nu_{\pi,i} + \varrho$

$$\left(i = \binom{n-1}{p-1} + \ldots + \binom{n-q-1}{p-1} + 1, \binom{n-1}{p-1} + \ldots + \binom{n-q-1}{p-1} + 2, \right.$$
$$\left. \ldots, \binom{n-1}{p-1} + \ldots + \binom{n-q}{p-1}\right).$$

We finally take $\mu_{1,i} = \mu_q$, for the indices i considered, so that

$$\mu_{1,i} \leqq q + \varrho = \nu_{1,i} + \varrho.$$

Then the vectors

$$[B^{(1,\mu_{1,i})}, B^{(2,\mu_{2,i})}, \ldots, B^{(p,\mu_{p,i})}],$$

and even their projections on $R^{(q)}$, are linearly independent, whereas these vectors are all lying in the linear subspace of R_N which is the direct sum of the spaces $R^{(1)}, R^{(2)}, \ldots, R^{(q)}$, and moreover $\mu_{\pi,i} \leqq \nu_{\pi,i} + \varrho$ for the indices i considered.

Applying the last result with $q = 1, 2, \ldots, n-p+1$ we immediately get the assertions of the lemma.

Proof of the theorem. The compound body K does not alter if we permute the bodies $K^{(1)}, K^{(2)}, \ldots, K^{(p)}$. We shall choose a definite arrangement of these bodies. Let $\mathfrak{S}$ denote an arbitrary permutation of p elements and put

$$f(\mathfrak{S}) = \prod_{(\nu_1, \nu_2, \ldots, \nu_p)} m_{\nu_1}^{(\mathfrak{S}1)} m_{\nu_2}^{(\mathfrak{S}2)} \ldots m_{\nu_p}^{(\mathfrak{S}p)},$$

where the product is extended over all sets of positive integers $(\nu_1, \nu_2, \ldots, \nu_p)$ with

$$1 \leqq \nu_1 < \nu_2 < \ldots < \nu_p \leqq n.$$

We further form the product of $f(\mathfrak{S})$ over all permutations $\mathfrak{S}$. One readily verifies that

$$\prod_{\mathfrak{S}} f(\mathfrak{S}) = \{\prod_{\pi=1}^{p} \prod_{\nu=1}^{n} m_\nu^{(\pi)}\}^{(n-1)!/(n-p)!}$$

Hence there exists a permutation $\mathfrak{S}$ for which

$$f(\mathfrak{S}) \leqq \{\prod_{\pi=1}^{p} \prod_{\nu=1}^{n} m_\nu^{(\pi)}\}^{P/p}.$$

By the initial remark, it is no loss of generality to suppose that $\mathfrak{S}$ is the identical permutation. So we may suppose that

$$\prod_{(\nu_1, \nu_2, \ldots, \nu_p)} m_{\nu_1}^{(1)} m_{\nu_2}^{(2)} \ldots m_{\nu_p}^{(p)} \leqq \{\prod_{\pi=1}^{p} \prod_{\nu=1}^{n} m_\nu^{(\pi)}\}^{P/p}. \tag{8}$$

We now consider the lattice points $A^{(\pi,\nu)}$ introduced earlier. For $\pi = 1, 2, \ldots, p$ the n vectors $A^{(\pi,1)}, A^{(\pi,2)}, \ldots, A^{(\pi,n)}$ are linearly independent.

Applying the lemma, with $m=n$, we obtain that there exist N sets of p positive integers,

$$\{\mu_{1,i}, \mu_{2,i}, \ldots, \mu_{p,i}\} \qquad (i=1, 2, \ldots, N)$$

say, such that the following two properties hold:

1. $\mu_{\pi,i} \leqq \nu_{\pi,i} \qquad (\pi=1, 2, \ldots, p; (i=1, 2, \ldots, N)$.
2. the N vectors

$$\Xi^{(i)} = [A^{(1,\mu_{1,i})}, A^{(2,\mu_{2,i})}, \ldots, A^{(p,\mu_{p,i})}] \tag{9}$$

in R_N are linearly independent.

The points $\Xi^{(i)}$ clearly have integral coordinates. So, by the property 2, the 2^N-hedron whose vertices are given by these points and their reflections in the origin, has volume $\geqq 2^N/N!$.

By (4), for all π and ν,

$$(1/m_\nu^{(\pi)}).\, A^{(\pi,\nu)} \in K^{(\pi)}.$$

Put

$$\{m_{\mu_{1,i}}^{(1)} m_{\mu_{2,i}}^{(2)} \ldots m_{\mu_{p,i}}^{(p)}\}^{-1}\, \Xi^{(i)} = \mathrm{H}^{(i)} \; (i=1, 2, \ldots, N).$$

Then

$$\mathrm{H}^{(i)} = [(m_{\mu_{1,i}}^{(1)})^{-1} A^{(1,\mu_{1,i})}, (m_{\mu_{2,i}}^{(2)})^{-1} A^{(2,\mu_{2,i})}, \ldots, (m_{\mu_{p,i}}^{(p)})^{-1} A^{(p,\mu_{p,i})}]$$

is a point of K, for $i=1, 2, \ldots, N$. So the 2^N-hedron with vertices $\pm\, \mathrm{H}^{(i)}$ is wholly contained in K. Hence we have

$$V(\mathrm{K}) \leqq (2^N/N!).\prod_{i=1}^{N} \{m_{\mu_{1,i}}^{(1)} m_{\mu_{2,i}}^{(2)} \ldots m_{\mu_{p,i}}^{(p)}\}^{-1}.$$

For each π, the successive minima $m_1^{(\pi)}, m_2^{(\pi)}, \ldots, m_n^{(\pi)}$ form a non-decreasing sequence. Hence, by the property 2, $m_{\mu_{\pi,i}}^{(\pi)} \leqq m_{\nu_{\pi,i}}^{(\pi)}$ $(\pi=1, 2, \ldots, p; i=1, 2, \ldots, N)$. Then it follows from (8) that we have

$$\prod_{i=1}^{N} \{(m_{\mu_{1,i}}^{(1)} m_{\mu_{2,i}}^{(2)} \ldots m_{\mu_{p,i}}^{(p)}\}^{-1} \geqq \{\prod_{\pi=1}^{p} \prod_{\nu=1}^{n} m_\nu^{(\pi)}\}^{-P/p}.$$

Finally, applying (3) with $\pi=1, 2, \ldots, p$, we get

$$V(\mathrm{K}) \cdot \{V(K^{(1)})\, V(K^{(2)}) \ldots V(K^{(p)})\}^{-P/p} \geqq 2^N\, 2^{-nP/p}/N!$$

This proves the theorem, with $c=1/N!$.

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